# ON THE GENERIC INSTABILITY OF MIXED STRATEGIES IN ASYMMETRIC CONTESTS

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Abstract

Although a mixed strategy can never be evolutionary stable in a truly asymmetric contest, examples show mixed strategies can satisfy the weaker criterion of neutral stability. This paper shows that such examples are rare, and generically, a mixed strategy is unstable. We apply our result to the battle of the sexes between males and females over the raising of offspring. Our result can be used to rule out specific forms of randomization in extensive form games, as we illustrate in the context of games with pre-play communication.

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## 1. INTRODUCTION

The concept of an evolutionary stable strategy (ESS) was developed by Maynard Smith and Price (1973) in the context of two-player symmetric games. Many games are however asymmetric, and a player can be in a number of possible roles, or information situations. The ESS concept was extended to asymmetric two-player games by Selten (1980). Selten symmetrized the game by adding a move of nature which determined the allocation of players to roles. We shall refer to this symmetrization of the underlying game as an asymmetric contest. Selten showed that the ESS concept is very restrictive in truly asymmetric contests, where two players are never in the same information situation: an ESS must be a strict Nash equilibrium of the underlying game. Since two important classes of games do not have strict Nash equilibria - games with only mixed strategy Nash equilibria, and games with a non-trivial extensive form - Selten's result implies that an ESS will not exist in the associated truly asymmetric contests.

Selten's negative result prompted an analysis of weaker notions of evolutionary stability. Selten proposed the limit ESS concept, and Thomas (1985) suggested the set-valued concept of an evolutionary stable set (ES set). The weakest concept which has been proposed in this context is Maynard Smith's (1982) neutrally stable strategy (NSS). Indeed, it is well known that a limit ESS is a NSS, as is any point in an ES set. These concepts have been used in a number of papers which analyze evolutionary stability in extensive form games, such as games with pre-play communication or infinitely repeated games.1 In such games pure strategy NSS exist while ESS do not, since the NSS concept is defined by a weak inequality rather than a strict one.

The question which remains open is the following: can a NSS be in mixed strategies in a truly asymmetric contest? The question is of importance since some games have only mixed strategy Nash equilibria. A much discussed biological example is the "battle of the sexes", between the males and females of the species, over parental investment in the care of offspring (see Dawkins, 1976; Hofbauer and Sigmund, 1988). This game is strictly competitive, and has a unique mixed equilibrium. It has been suggested that the mixed equilibrium is stable (Hofbauer and Sigmund, 1988), and there also exist examples of games where the mixed equilibrium is neutrally stable. For instance, a mixed strategy equilibrium of any zero-sum game is a NSS, and it is also possible to to find a mixed NSS in some strictly competitive games which are not zero-sum. The question is, to what extent do these examples generalize?

The answer of this paper is: very little. Our main theorem, which is presented in section 2, shows that a mixed strategy is a NSS of the asymmetric contest only if the payoffs in the game satisfy a linear equality, which is not possible generically. We use this result to discuss the battle of the sexes in section 3, and show that our negative result applies to this game.

The NSS concept has mainly been applied to extensive form games, where the normal form necessarily has payoff ties. Although we do not provide a general analysis of asymmetric contests where the underlying game has a non-trivial extensive form, we suggest that our arguments can be employed to rule out specific forms of randomization in such games. We illustrate this in section 4, by analyzing cheap talk games, where players exchange costless messages before playing a generic normal form game. The cheap talk game does not even have a generic extensive form (since payoffs are equal at different end-points of the game tree). Nevertheless, we show that the outcome of an NSS of the cheap talk game is a pure Nash equilibrium of the underlying game, and is hence non-random. This rules out the possibility of players using cheap talk to play a jointly-controlled lottery.

## 2. THE ANALYSIS

Our asymmetric game set up follows van Damme (1991) closely. Consider a situation where two players are randomly drawn from a single population and matched to play a bi-matrix game. Each of these players can be in one of several information situations, which is a complete description of the state of the player, and may include some information regarding the other player’s state. Let U be the (finite) set of information situations, and let C denote the finite set of choices available at $u \varepsilon U$, with cardinality $|C|$. A contest is the collection $<\{u,v\},C ,C ,A ,A >$, where u and v are the information situations of player 1 and 2 respectively, C and C are the sets of choices, and A is the payoff matrix of the player in role u, and A is the payoff matrix of the player in role v.

A local strategy at u is an element of DC , and will be denoted by b . A behavior strategy, b, is a vector of local strategies, one for each information situation. Write B for the set of local strategies at u, and B for the set of behaviour strategies.

The two players are randomly allocated to information situations by a symmetric probability measure over UxU, p, which is consistent (see van Damme, 1991). Let p denotes the probability of a uv contest with player 1 in situation u. Let p := Sp be the probability of player 1 being in information situation u, and assume that p > 0 for all u. A game is said to be truly asymmetric if the two players are never in the same information situation, i.e. if p = 0 for all u. In this paper we shall concern ourselves only with truly asymmetric games. A truly asymmetric game can also be seen as a $|U|$ person game with one player for each information situation. This is called the agent normal form of the game.

Let b and b′ be two behavior strategies, of player 1 and player 2 respectively. The ex-ante expected payoff of player 1 is given by:

$$A(b,b') := S\ S\ \ p\ \ b\ A\ \ b' \qquad (2.1)$$

Since the probability measure is symmetric, the expected payoff of player 2 equals $A(b',b)$, where this is defined as in (2.1). Hence the asymmetric contest is given by the symmetrized game where each player has strategy set B and payoff function given by (2.1).

A behavior strategy $b \in B$ is said to be a Neutrally Stable Strategy (NSS) if for all $b' \in B$, (2.2) and (2.3) apply:

$$A(b,b) \geq A(b',b) \qquad (2.2)$$

$$A(b,b) = A(b',b)\ 6\ A(b,b') > A(b',b') \qquad (2.3)$$

A strategy will be called unstable if it fails to be neutrally stable.

The strategy b is an Evolutionary stable strategy (ESS) if the inequality in (2.3) is strict for all $b'=b$.

Although the ESS and NSS concepts are defined in static terms, they are closely related to concepts of dynamic stability. Consider a polymorphic population where different fractions of the population play different strategies, and where the rates of increase of various strategies in the population are governed by the replicator dynamics. Taylor and Jonker (1978) analyzed the dynamics in the situation where individuals are restricted to pure strategies, and showed that if a strategy is an ESS, then the corresponding state is asymptotically stable. Thomas (1985) showed that a NSS is Lyapunov stable. The converses of these statements are not true if individuals in the polymorphic population are restricted to playing pure strategies. However, if individuals can play mixed strategies, a version of the converse applies (see Hines, 1980; Bomze and van Damme, 1992).

We now present the basic idea of Selten's result showing that an ESS in a truly asymmetric contest is a strict Nash equilibrium of the agent normal form.

Let b = (b ,b ,...,b ) be a behavior strategy, and let b′ be an alternative best response to (b ,...b ). Consider the strategy b′ = (b′,b ,b ,..,b ), which differs from b only with regard to the choice in role 1. Since b′ is a best response to (b ,...b ), we have:

A(b′,b) = A(b,b) (2.4)

Further, since p = 0, it is easily verified that:

A(b′,b′) = A(b,b′) (2.5)

(2.4) and (2.5) imply that b is not an ESS, but have no implication for the neutral stability of b. Indeed, NSS often exist in extensive form games which do not have strict Nash equilibria.

The main theorem of this paper applies to generic truly asymmetric contests. Let the collection <U, {C } ,p> be given and fixed. The specification of the truly asymmetric contest is complete once we specify payoffs to each pair of actions in each contest. The number of such payoffs is n := S S |C |x|C |. Hence each point in R in conjunction with the collection <U, {C } ,p> completely specifies a truly asymmetric contest. We say that a property $\alpha$ applies for almost all truly asymmetric contests if given any collection <U, {C } ,p>, the set of points in R such that $\alpha$ fails to apply is closed set of Lebesgue measure zero in R . We are now in a position to state the main result of our paper.

Theorem In almost all truly asymmetric contests, any NSS is a pure strategy Nash equilibrium.

Corollary For almost all truly asymmetric games, an NSS of $\Gamma$ is a strict Nash equilibrium of $\Gamma$, since pure strategy Nash equilibria are generically strict.

We prove the theorem via four lemmata. The first of these shows that if a behavior strategy b is a NSS, then it must be stable against mutants which vary their behavior only at a pair of information situations. The second lemma shows that if a single component, b , is mixed, then there must be a second compnent b which is also mixed. Using these two lemmata, we restrict attention to mutants which vary their behavior only at the information situations u and v, and which use pure strategies which are in the supports of b and b . We then show that the NSS conditions reduce to a set of linear equality constraints upon the payoff matrices

A and A , which cannot be satisfied generically.

Define the local game at uv induced by b as follows: the game consists of a pair of information situations, (u,v), pure strategy sets C and C respectively, and payoff matrices A , A , which are defined by:

A := p A + S p A b h (2.6)
A := p A + S p A b h (2.7)

where b is written as a column vector and h and h are row vectors of ones, (1,1,....,1), of dimension |C | and |C | respectively.

(2.6) shows that A is probability weighted sum of A and a second matrix, which gives u's payoff in contests uw (w = v) given local strategies b . This second matrix has constant rows since the payoff in contests uw does not depend upon the choices made by the v-indivdual.

In the symmetrized version of the local game at uv defined by b, each player has the strategy set B := B xB . If player 1 adopts strategy b" and 2 the strategy b' , the payoff to 1 is given by:

A(b" ,b' ;b,u,v) := b"A b' + b"A b' (2.8)

Neutral stability in the symmetrized version of the local game at uv defined by b can be defined as in (2.2) and (2.3).

A strategy b will be called pairwise neutrally stable (PNSS) if for every pair (u,v), the component (b ,b ) is a NSS of the local game defined by b at uv.

Lemma 2.1 provides a quasi-local characterization of NSS in truly asymmetric games. For proving our theorem, we require the "only if" part of this lemma suffices. We state and prove the full equivalence of the lemma since it may be of some independent interest.

Lemma 2.1 Let Γ be a truly asymmetric game. b is a NSS of Γ if and only if b is a PNSS of Γ.

Proof: Note first that due to the equivalence between Nash equilibria of the agent normal form and the symmetrized game:

A(b,b)>A(b',b) A b'ε B 46 A(b ,b ;b,uv) > A(b' ,b ;b,uv)

A b' ε B , A (u,v) ε UxU (2.9)

Hence we can let b be a Nash equilibrium, and restrict attention to mutants b' which are best responses to b, so that A(b,b) = A(b',b). Consider the expression:

$$A(b,b';b,uv) - A(b',b';b,uv) = p\ (b - b')A\ b' + p\ (b - b')A\ b'$$
$$+ S\ p\ (b - b')A\ b + S\ p\ (b - b')A\ b$$

The proof of "only if": Let the expression in (2.10) be strictly negative so that b is not an NSS of the local game at uv induced by b. Define b′ to coincide with b at all information situations w = u,v, and to play b′ and b′ at u and v respectively. It is clear that A(b,b′) - A(b′,b′) equals the expression in (8), so that b is not an NSS.

To prove "if", re-write (2.10) as :

$$A(b,b';b,u,v) - A(b',b';b,u,v) = p\ (b - b')A\ b' + p\ (b - b')A\ b'$$
$$- p\ (b - b')A\ b + p\ (b - b')A\ b \quad (2.11)$$
$$+ S\ p\ (b - b')A\ b + S\ p\ (b - b')A\ b$$

Take the sum of (2.11) over all distinct u,v in UxU, to get:

u$v v

$$= S\ S\ p\ (b - b')A'\ b' + 2(|U|-2)\ S\ S\ p\ (b - b')A'\ b$$

$$= 2[A(b,b') - A(b',b')] - 2(|U|-2)[A(b,b) - A(b',b)] \quad (2.12)$$

Re-arranging, we have:

$$A(b,b') - A(b',b') = S/2 + (|U|-2)[A(b,b) - A(b',b)] \quad (2.13)$$

If b is a PNSS, each term in the summation (S) is positive. Since A(b,b) = A(b′,b), the second term is zero and b is an NSS. p

Lemma 2.2 If b is a Nash equilibrium such that the component at u, b , is a non-degenerate mixed strategy, then for almost all games, there exists a v different from u with p >0 such that b is also a non-degenerate mixed strategy.

Proof Suppose not so that b = c , a pure strategy, for all w such that p >0. Let c , c′ belong to the support of b , which implies:

$$S\ p\ (c - c')A\ c = 0 \quad (2.14)$$

(2.14) defines a hyperplane, of Lebesgue measure zero in the space of payoffs. The set of payoffs such that there is mixing only at one information situation is the finite union of such hyperplanes, and hence of Lebesgue measure zero. p

Let b be a mixed strategy NSS. By lemma 2.2, there is a pair of information situations, (u,v), such that b and b are mixed. Let C(b ) and C(b ) denote the supports of b and b . Note that if (b ,b ) is a NSS of the local game defined at uv, then (b ,b ) is a NSS of the local game where the player at u (resp. v) is restricted to mixed strategies which only use those pure strategies which are in C(b )(resp. C(b )). Consider the restricted local game <C(b ), C(b ), A , A >, where A is the restriction of A to C(b )xC(b ) and A is the restriction of A to C(b )xC(b ). Let B denote the set of behavior strategies in the restricted local game. Note that b and b are completely mixed strategies in the resticted game, i.e. they are in the interior of DC(b ) and DC(b ) respectively. Let A and A denote the restricted versions of the payoff matrices of the contest at (u,v).

Lemma 2.3 If b is a mixed NSS and b′ is any mutant strategy in B , then:

b A b = b′A b and b A b = b′A b (2.15)

b A b′ + b A b′ = b′A b′ + b′A b′ (2.16)

Proof Since b is a best response to b which is completely mixed in the restricted game, every strategy in DC(b ) has the same payoff against b . The same argument applies to the v-individual, hence (2.15) applies.

Since b is a NSS of the restricted game and (2.15) applies, the NSS condition implies the inequality:

b A b′ + b A b′ > b′A b′ + b′A b′ (2.17)

We claim that (2.17) must hold with equality for b′ ε B . To see this, assume the contrary, so that there exists b′ such that the inequality (2.17) is strict. Since b , b are both in the interior of DC(b ) and DC(b ) respectively, there exist scalars l >0 and l < 0 such that b" := (1-l )b + l b′ belongs to DC(b ), b" := (1-l )b + l b′ belongs to DC(b ). A(b,b";b,u,v) - A(b",b";b,u,v) is given by:

(b -b")A b" + (b -b")A b" = l l [(b -b′)A b′ + (b -b′)A b′]

+ l (1-l )(b -b′)A b + l (1-l )(b -b′)A b

(2.15) implies that the last two terms on the right vanish, so that:

(b -b")A  b" + (b -b")A  b" = l l [(b -b')A  b' + (b -b')A  b']  (2.19)

Since the sign of (2.19) is the opposite of that of (2.17), this implies that (2.17) must hold with equality, i.e. (2.16) is valid for all b' .        p

Let  D  :=  A    +  (A  ) . D is the sum of restricted payoff matrices of the local game at uv defined by b.

Let  F  :=  A    +  (A  ) . F is the sum of restricted payoff matrices of the contest uv.

If  b   is a mixed strategy NSS of the local game at uv defined by b, then by lemma 2.2, D and F are at least 2x2. Consider 2x2 submatrices of D (resp. F). Index the rows by h,i and columns by j,k, and write d   (resp. f  ) for the corresponding entry in the submatrix of D (resp F).

Lemma  2.4  If  b   is a mixed strategy NSS of the local game, then every 2x2 sub-matrix of F satisfies: f   + f   = f   + f  .

Proof  Let e  and e  be two-pure strategies for u in the restricted game, and let  e   and  e  be two pure strategies for v. Write (2.16), letting b'  be each of the four possible pure combinations in turn, giving:

b A  e  + b A  e  = e A  e  + e A  e  = d          (2.20)

b A  e  + b A  e  = e A  e  + e A  e  = d          (2.21)

b A  e  + b A  e  = e A  e  + e A  e  = d          (2.22)

b A  e  + b A  e  = e A  e  + e A  e  = d          (2.23)

Subtracting  (2.21)  from  (2.20) yields [b A  e  - b A  e ] on the left-hand side.  Similarly,  subtracting  (2.23)  from  (2.22)  yields  the  same [b A  e  - b A  e ] on the left hand side. This implies that:

d   + d    = d   + d                          (2.24)

Since  h,i,j  and k were arbitrarily chosen, (2.24) holds for every  2x2 sub-matrix of the matrix D.

Recall  (2.6)  and  (2.7), which show that A   is the sum of A   and a matrix with  constant  rows.  Hence  D is the sum of F, a matrix with constant rows, and a matrix with  constant columns. Hence, for every 2x2 sub-matrix of F, we have:

f   + f    = f   + f                          (2.25)    p

The proof of theorem is now straight-forward. If b is a mixed strategy NSS of $\Gamma$, lemma 2.4 implies the payoffs of $\Gamma$ satisfy a linear equality, and the set of payoffs satisfying this is a closed set of Lebesgue measure zero in R . p

Remark 1. Consider a truly asymmetric game with two roles, where there is no asymmetry in payoffs, but only in roles. In this case A = A , but players may condition their actions upon the role they fill (see van Damme, 1991 or Samuelson, 1991 for a discussion of such games). The theorem applies to such games as well, since within the class of symmetric matrices, the set of those satisfying the condition of lemma 2.4 are of measure zero.

## 3. THE BATTLE OF THE SEXES

Our results clarify an ongoing controversy in the biological literature, on the Battle of the Sexes between the males and females of the species, over the care of offspring (see Dawkins, 1976). The females of a species usually produce a few large gametes whereas males produce many small ones, implying that females have less options of deserting the offspring, whilst a male can desert and increase their offspring by gaining a new mate. The defence of females against this is to be coy, i.e. to insist on a long and costly engagement period before copulation. In a population of coy females, males are better off being faithful thereby avoiding a second engagement cost.

Let V be the value of the offspring to each parent, let 2C be the total cost of raising the offspring, and let c be the engagement cost to each partner. The male can be PHILANDERING or FAITHFUL whilst the female can be FAST or COY. The payoffs to these strategies are set out in G1. As in van Damme (1991), we assume that:

$$0 < c < C < C+c < V < 2C \tag{3.1}$$

This implies that G1 has a unique Nash equilibrium in mixed strategies. Since both males and females belong to the same species (and hence to a single population), the asymmetric conflict corresponding to G1 is clearly relevant for analysis. Since the eight payoffs in G1 only depend upon three parameters, c, C and V, so that we cannot directly apply the theorem in the preceding section. Nevertheless, lemma 2.4 makes clear that the mixed strategy equilibrium in not an

NSS, since the condition of the lemma is only satisfied if:

$$2[V-C-c] + 2[V-C] = 0 + 2[V-C] \qquad (3.2)$$

This requires $V = C + c$, which contradicts our assumption (3.1). Hence we can conclude that the mixed strategy equilibrium is unstable in the Battle of the Sexes. To take illustrative numbers, if $V = 5$, $C = 3$ and $c = 1$, the mixed strategy equilibrium has the male being FAITHFUL with probability 1/2 and the female COY with probability 3/4. This mixed strategy can invaded by the pure strategy (FAITHFUL,COY), and can also be invaded by the pure strategy (PHILANDERING,FAST).

Our analysis contrasts with the results of Hofbauer and Sigmund (1988). They assume that males and females belong to distinct populations and analyze the pure strategy replicator dynamics associated with G1, and show that the mixed strategy equilibrium is asymptotically stable. Hofbauer and Sigmund also suggest a "weak ESS" criterion for the two population case - a strategy b = (b ,b ) is a weak ESS if b is a Nash equilibrium and if for any b′ = (b′,b′):

$$A\ (b',b') > A\ (b\ ,b')\ 6\ A\ (b',b') < A\ (b',b\ ) \qquad (3.3)$$

where A is the payoff in role i, i = 1,2.

The weak ESS criterion requires that there is no mutant which does better in both roles. Since the mixed strategy equilibrium satisfies this criterion, it is a weak ESS.

This divergence in results stems from our different assumptions. Our analysis considers a single population, members of which may ex ante be in either role, whilst Hofbauer and Sigmund assume distinct populations. The two-population specification is more appropriate for interaction between different species, whilst the one-population specification may be more appropriate for analyzing male-female interaction within the same species.[2] This distinction is also relevant for the the validity of re-scaling payoffs, a device which is often adopted in this literature. Hofbauer and Sigmund show that by making two positive affine transformation of the payoffs, one for each role, one can convert G1 into a zero-sum game. Such a rescaling is only valid in the two-population case, and is not valid in the asymmetric contest specification - since a player may with equal probability be in either role, one can only make a single transformation which applies to all payoffs, in either role.

## 4 GAMES WITH COSTLESS PRE-PLAY COMMUNICATION

The main theorem of this paper applies to truly asymmetric contests where the underlying game is a generic normal form game. The NSS concept (and related weak concepts) have mainly been applied to extensive form games. The normal form of such games will have payoff ties, since a player's actions at an unreached information set does not affect payoffs. Does our theorem have any implication for such games? In answer to this question we offer, without proof, the following conjecture. Let $\Gamma$ be an extensive form game with two roles and no chance moves, and let b be a behavior strategy in the associated asymmetric contest. Let the outcome of b be the probability distribution over the terminal nodes of $\Gamma$. The conjecture we propose is the following: for almost all assignments of payoffs to the terminal nodes of $\Gamma$, if b is a NSS, the outcome of $\Gamma$ is deterministic. This conjecture allows that that b may prescribe randomization at some information sets; however, if b prescribes randomization at an information set u, then u is not reached when b plays itself. This conjecture could possibly be proved along the lines of the proof of our theorem in section 2. First, if the outcome of b is random, this must be due to randomization in both roles, if there are no payoff ties at the terminal nodes of $\Gamma$. Such randomization must imply a linear equality constraint on the payoffs at these terminal nodes similar to that in lemma 2.4. However, rather than attempt to prove this conjecture in general, we consider a specific extensive form game, and show the implications of our analysis.

Consider a cheap talk game, where players exchange costless messages and then play a (generic) normal form game. These games have been a focus of recent research on equilibrium selection - Sobel (1993) provides an excellent survey. The normal form of a cheap talk game has many payoff ties, for two distinct reasons. First, these games have a non-trivial extensive form, so that actions at unreached information sets do not affect payoffs. Second, messages do not affect payoffs directly, so that if a player chooses different messages but takes the same action in the underlying game, he gets the same payoff. For this reason such games do not have any ESS, while neutrally stable strategies often exist. Our main purpose in analyzing such games is to show that our results in section 2 allows us to place significant restrictions upon neutrally stable outcomes - specifically, we can show that the outcomes in terms of the underlying game must be non-random.

Let the underlying game G be a truly asymmetric game with two roles, where C and C are the set of actions of each player, and A and A are the associated payoff matrices. Let C = C xC . A behavior strategy in the asymmetric contest defined by G is a pair b = (b ,b ) where b ε DC , i = 1,2. The payoff of strategy b against b'is given by:

$$A(b,b') := [A (b ,b') + A (b',b )]/2 = [b A b' + b A b']/2 \qquad (4.1)$$

The cheap talk game, G*, associated with G, is as follows. Players are randomly drawn from a large population, and paired. Each player is assigned a role, i.e. role 1 or role 2. Each player sends a message from a finite set of messages, M, which contains at least two elements. After these messages are exchanged, the players play G. Payoffs in the cheap talk game, G*, do not depend upon the messages exchanged and depend only upon the actions taken in G.

A local behavior strategy in role i, x , is a pair (σ ,Θ ) where σ ε DM, and Θ : MxM——cDC . A behavior strategy in the cheap talk game, x, is a pair (x ,x ). Let σ = (σ ,σ ) and Θ = (Θ ,Θ ). Let S denote the set of behavior strategies in G*. The expected payoff of strategy x against x'=(σ',Θ') is given by:

$$u(x,x') := (1/2)\ S\ A\ [\Theta\ (m\ ,m\ ),\Theta'(m\ ,m\ )]\sigma\ (m\ )\sigma'(m\ ) \qquad (4.2)$$
$$+\ (1/2)\ S\ A\ [\Theta\ (m\ ,m\ ),\Theta'(m)]\sigma\ (m\ )\sigma'(m\ )$$

G* also defines an asymmetric contest. Further, since actions after unsent messages do not affect payoffs, and since players are indifferent between messages which induce the same outcome, G* has no strict Nash equilibria, and hence by Selten's (1980) theorem, has no ESS. However, if G is a compatible game, i.e. a game with a strict Nash equilibrium which is also efficient so that the sum of payoffs to the two roles is maximized at this action combination, it can be shown that G* possesses a neutrally stable strategy. An example of such a strategy is one which sends an arbitrary message and plays the efficient action pair after every message. Hence the NSS concept allows existence whilst the ESS concept does not.

Our results in this section invoke the following regularity assumption on the payoffs in G, which will be satisfied generically. If c = (c ,c ) ε C xC , let A(c) := A (c ,c ) + A (c ,c ) be the sum of payoffs to the two roles at c.

REGULARITY ASSUMPTION : The payoffs in the underlying game G satisfy:

R1 No payoff ties: A (c) = A (c′) for i = 1 or 2 6 c = c′.

R2 Let c,c′,c",c"" ε C xC , which need not be all distinct.

If A(c) + A(c′) = A(c") + A(c""), then c = c′ = c" = c"".

If x = (σ,Θ) is a behavior strategy, let the outcome of x be the probability distribution over C xC which is induced by the pair (x,x). Define also the following sets:

supp(σ ) := {m ε M: σ (m) > 0}

supp(σ) := supp(σ ) x supp(σ )

Proposition If x = (σ,Θ) is a NSS of G* and G satisfies the regularity assumption, the outcome of x is a pure strategy Nash equilibrium of G, i.e:

i) Θ(m) is a pure strategy Nash equilibrium of G A m ε supp(σ).

ii) Θ is a constant function on supp(σ).

Proof of (i) We show first that if m ε supp(σ), then Θ(m) is an NSS of G. Suppose to the contrary that for some m* ε supp(σ), Θ(m*) = b is not be an NSS of G. This implies that there exists b′ε DC xDC such that (4.3) or (4.4) apply:

A(b′,b) > A(b,b) (4.3)

or A(b′,b) = A(b,b) and A(b′,b′) > A(b,b′) (4.4)

Let y = (σ,Θ′) where Θ′(m) = Θ(m) A m = m*, and Θ(m*) = b′. y sends the same messages as x, and takes the same actions after all message pairs except m*.

A(y,x) - A(x,x) = σ(m*)[A(b′,b) - A(b,b)] (4.5)

A(y,y) - A(x,y) = σ(m*)[A(b′,b′) - A(b,b′)] (4.6)

Since σ(m*) > 0 this implies that x is not a NSS of G*. However, since G satisfies the regularity assumption, lemma 2.4 implies that any NSS of G is a pure strategy Nash equilibrium of G. Hence if m ε supp(σ), Θ(m) is a pure strategy Nash equilibrium of G. p

Proof of (ii) If supp(σ ) has two elements, m and m′, whilst supp(σ ) has one (m ), i must get the same payoff from m and m′, so that:

A (Θ(m ,m )) = A (Θ(m′,m ) (4.7)

which, by the regularity assumption R1, implies that Θ is constant on supp(σ).

Consider the case where supp(σ ) has two or more elements, for i = 1,2. since the payoffs to role i from any message in supp(σ ) is the same, this defines a mixed strategy equilibrium in messages. Let m , m′ ε supp(σ ), i= 1,2. By lemma 2.3 we have:

A(Θ(m ,m )) + A(Θ(m′,m′)) = A(Θ(m′,m )) + A(Θ(m ,m′)) (4.8)

By part (i) of the proposition, Θ(m) ε C xC for any m ε supp(σ). Hence if the regularity assumption R2 is satisfied, (4.8) implies that Θ is constant on supp(σ). p

This proposition shows that one can use our main theorem to impose significant restrictions upon outcomes even in non-generic extensive form games. Note the the proposition does not rule out some types of randomization. First, since all messages are equally costless players may randomize across messages in a NSS, provided that they take the same actions after all these messages. Second, the proof shows that there need not be any restriction upon Θ(m) if m is an unsent message, so that the probability σ(m) = 0. Hence a NSS of G* can well play random actions after such unsent messages.[3]

The significance of part (ii) of the proposition is that it rules out the possibility of a strategy using cheap talk to play a jointly controlled lottery over the pure Nash equilibria of G. If the underlying game G fails to satisfy the regularity assumption, a NSS of G* may play such a jointly controlled lottery over inefficient Nash equilibria of G. Such a strategy is often resilient, and jointly controlled lotteries have provided some counter-examples (see Kim and Sobel, 1994; Bhaskar, 1994; Schlag, 1994) to the proposition that "cheap talk entails efficiency". Our proposition shows that these examples hinge upon the G violating the regularity assumption.

We conclude this section with an example showing the conditions under which jointly controlled randomization is possible. We consider the asymmetric Hawk-Dove game (see Dawkins, 1976; van Damme 1991). Two animals of a species are contesting a resource. An animal can either be the OWNER of the resource, or the INTRUDER. The value of the resource to the OWNER is V , and the value to the INTRUDER is V , with V > V > 0. In either role, an animal has two possible strategies H (for hawk) or D (for dove). If both roles choose H, each incurs a cost C, from the conflict, and gets the resource with probability 1/2. We assume that :

$$C > V/2 > V/2 > 0 \qquad (4.9)$$

(4.9) implies that the game G2 has two pure strategy Nash equilibria, HD and DH, where the first strategy in the pair denotes the choice of the OWNER. Correspondingly, the associated asymmetric contest has two ESS (and NSS): the bourgeois strategy HD where the OWNER chooses H and the INTRUDER chooses D, and the paradoxical strategy DH, where the OWNER chooses D and the INTRUDER H. If V > V , the bourgeois strategy is more efficient, from the point of view of the species, than the paradoxical strategy; however, this is not an evolutionary or game theoretic argument against the latter.

We now augment this game by allowing the animals to exchange costless messages from the set {1,2}, before playing G2. The augmented game has no ESS, but always has a NSS. Consider the following strategy LOTTERY which sends both these messages with equal probability. If the realized messages coincide, i.e. if they are (1,1) or (2,2), the OWNER plays H and the INTRUDER D. If the messages differ, so that they are (1,2) or (2,1) the OWNER plays D and the INTRUDER H. The payoffs to message combinations is shown in Fig. 3. LOTTERY plays a jointly controlled lottery over the pure Nash equilibria of G2, HD and DH. LOTTERY is a Nash equilibrium for all relevant values of the parameters, V , V and C. However, proposition 1 makes clear that it can be a NSS of the game with cheap talk only if V = V , i.e. if there is no asymmetry in payoffs, only an asymmetry in roles. If V > V , and if the animals are using payoff irrelevant signals to coordinate, in the manner of LOTTERY, it can be verified that this can be invaded by a strategy which sends the same message in either role and coordinates on HD. In other words, the bourgeois strategy can evolve from a population playing LOTTERY while the paradoxical strategy cannot.

One can go further and show that if $V > V$, pre-play communication destablizes the paradoxical strategy, which fails to a NSS. The unique NSS then corresponds to the bourgeois strategy. This however requires us allow for the possibility that the animals sometime mis-interpret signals or make mistakes in sending them. This is beyond the scope of this paper - we refer the reader to Bhaskar (1994) for an analysis of pre-play communication incorporating noise.

## 5. CONCLUDING COMMENTS

This paper has shown that randomized strategies do not generically satisfy even the weak criterion of neutral stability in truly asymmetric conflicts. This has implications for biological games such as the battle of the sexes. Our result can also be used to rule out randomized outcomes in asymmetric conflicts with an extensive form, as we illustrate in the context of games with pre-play communication.

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1 See Fudenberg and Maskin (1990) and Binmore and Samuelson (1992) for analyses using the NSS concept in repeated games, and section 4 of this paper for a discussion of games with costless pre-play communication.

2 Hofbauer and Sigmund (1988, p141) recognize this problem, and stress that their analysis "not only assumes asexual reproduction, but also that the phenotypes of one population are unrelated with that of the other. For games between different species, this is fine; but for the Battle of the Sexes...it does not look convincing."

3 Such randomization after unsent messages may sustain a NSS which plays an inefficient outcome, as Bhaskar (1994) shows. However, this may be eliminated if one assumes that players make "mistakes" so that all messages are sent with positive probability.

**FEMALE**

| **MALE** | | COY | FAST |
|---|---|---|---|
| | FAITH. | V-C-c, V-C-c | V-C, V-C |
| | PHIL. | 0, 0 | V, V-2C |

**G1** Battle of the Sexes

**INTRUDER**

| **OWNER** | | H | D |
|---|---|---|---|
| | H | $1/2(V_O-C)$, $1/2(V_I-C)$ | $V_O$, 0 |
| | D | 0, $V_I$ | $1/2\ V_O$, $1/2\ V_I$ |

**G2** The Hawk-Dove Game with Ownership

| | | INTRUDER 1 | INTRUDER 2 |
|---|---|---|---|
| OWNER | 1 | $V_O$ / 0 | 0 / $V_I$ |
| OWNER | 2 | 0 / $V_I$ | $V_O$ / 0 |

**Fig 3:** The Strategy LOTTERY